8.05 Review Notes

Information on the formula sheet is not usually reproduced here...

The First Bits of the Course...

- For a free particle

$$\boxed{\hat{H} = \frac{\hat{p}^2}{2m} + V(x) = \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)}$$

- General features of wavefunctions
  - The *ground state* must be *even*.
  - The number of nodes indicates how "high" the state is.
- To incorporate the fact a particle decays as $\exp\left(-t / \tau\right)$, add $-i\hbar / 2\tau$ to the potential.
- The particle flux is given by $\boxed{J(x,t) = \frac{\hbar}{m}\operatorname{Im}\left(\psi^* \frac{\partial \psi}{\partial x}\right)}$ with $\dot{\rho} + J' = 0$. To prove, write an expression for $\dot{\rho} = \frac{\mathrm{d}}{\mathrm{d}t}(\psi\psi^*)$ and simplify with a complex-conjugated SE. Integrating the conservation law over all space, we end up with the fact total probability is conserved. For a fluid, $J = \rho v$ which gives us a nice definition of quantum velocity.
- $\left[\hat{x}, \hat{p}\right] = i\hbar$
- General tips and tricks with Dirac Notation
  - A crucial step in many derivations is that

$$\begin{aligned}\left\langle x \mid \hat{p} \mid p\right\rangle &= p\left\langle x \middle| p\right\rangle \\ &= p\frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar} \\ &= -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar} \\ &= -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\left\langle x \middle| p\right\rangle\end{aligned}$$

    And similarly that

$$\begin{aligned}\langle p \mid \hat{x} \mid x\rangle &= x\langle p|x\rangle \\ &= x\frac{1}{\sqrt{2\pi\hbar}}e^{-ipx/\hbar} \\ &= i\hbar\frac{\mathrm{d}}{\mathrm{d}p}\langle x|p\rangle\end{aligned}$$

- When working out expressions like $\langle x \mid \hat{p} \mid \psi\rangle$, write it as $\int\langle x \mid \hat{p} \mid p\rangle\langle p \mid \psi\rangle \mathrm{d}p$.
- To find $\langle x \mid \hat{p} \mid y\rangle$, insert the identity into $\langle x \mid \hat{p} \mid \psi\rangle$ and compare.

- When showing that $e^{\hat{A}}|\alpha\rangle$ is an eigenstate (of x, say), easiest way to do it is

$$\hat{x}e^{\hat{A}}|\alpha\rangle = e^{\hat{A}}\left(e^{-\hat{A}}\hat{x}e^{\hat{A}}\right)|\alpha\rangle = e^{\hat{A}}\left(\hat{x}+\left[\hat{x},\hat{A}\right]\right)|\alpha\rangle$$

[This uses the fact that $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B}+[\hat{A},\hat{B}]$, quoted on the formula sheet].

- To find the Fourier Transform of a function like xe^{ipx}, eliminate the x by expressing it as a derivative of the exponential.
- The function of the doubly differentiated delta function is to "pick out" the double derivative.
- To find the maximum and/or minimum value of an operator $\hat{A}$, consider a normalised eigenvector ψ and realise that $\langle \psi \mid \hat{A} \mid \psi\rangle = a\langle\psi|\psi\rangle$. Then, write $\hat{A}$ in two ways that makes $\langle \psi \mid \hat{A} \mid \psi\rangle$ a norm, and realise it must therefore be greater than 0 (for example, $\langle \psi \mid \hat{a}^\dagger\hat{a} \mid \psi\rangle = \left\|\hat{a}|\psi\rangle\right\| \geq 0$ and $\langle \psi \mid 1-\hat{a}\hat{a}^\dagger \mid \psi\rangle$).
- For a free particle, the wavelength is given by

$$\frac{\hbar^2k^2}{2m} = E - V_{eff}$$

- Translation operators in QM

$$\boxed{\mathcal{U} = e^{-ix_0\hat{p}/\hbar}} \qquad \langle x|\hat{\mathcal{U}}|\psi\rangle = \psi(x-x_0) \qquad \hat{\mathcal{U}}^\dagger\hat{x}\hat{\mathcal{U}} = x + x_0$$

$\boxed{\mathcal{U} = e^{i\phi\hat{J}_z}}$ rotates the system by ϕ about the z-axis

- The postulates of QM:
  - At each instant, the state of a physical system is represented by a **ket** $|\psi\rangle$ in the space of states.

- Every observable attribute of a physical system is described by an Hermitian operator that acts on the kets that describe the system.
- The only possible result of the measurement of an observable A is one of the eigenvalues of the corresponding operator A.
- When a measurement of an observable A is made on a generic state $\left|\psi\right\rangle$, the probability of obtaining an eigenvalue a_n is given by the square of the inner product of $\left|\psi\right\rangle$ with the eigenstate $\left|a_n\right\rangle$ – $\left|\left\langle a_n \middle| \psi\right\rangle\right|^2$.
- Immediately after the measurement of an observable A has yielded a value a_n, the state of the system is the normalised eigenstate $\left|a_n\right\rangle$.
- The time-evolution of a quantum system preserves the normalisation of the associated ket. The time evolution of the state of a quantum system is described by $\left|\psi(t)\right\rangle = \hat{U}(t,t_0)\left|\psi(t_0)\right\rangle$ where U is unitary.

Uncertainty

- If $\left[\hat{A},\hat{B}\right] = 0$, then A and B are compatible – simultaneous eigenfunctions.
- The **complete set of commuting observables** is one in which each basis state is specified by a unique set of eigenvalues.
- For incompatible observables, the generalised uncertainty principle states that

$$\left(\Delta A\right)^2\left(\Delta B\right)^2 \geq \left(\left\langle \psi \mid \tfrac{1}{2i}[\hat{A},\hat{B}] \mid \psi\right\rangle\right)^2$$

We note that $\frac{1}{2i}[\hat{A},\hat{B}]$ is Hermitian and so has real expectation values. Thus, the RHS is always real and positive. Note that even if $[\hat{A},\hat{B}] \neq 0$, the expectation value of $i[\hat{A},\hat{B}]$ might still be 0.

- To prove the generalised uncertainty principle, consider

$$\left|f\right\rangle = \left(\hat{A} - \left\langle\hat{A}\right\rangle\right)\left|\psi\right\rangle$$
$$\left|g\right\rangle = \left(\hat{B} - \left\langle\hat{B}\right\rangle\right)\left|\psi\right\rangle$$

And

$$\langle f \mid f\rangle\langle g \mid g\rangle \geq \left|\langle f \mid g\rangle\right|^2$$

$$(\Delta A)^2(\Delta B^2) \geq \cancel{\left(\frac{\langle f \mid g\rangle + \langle g \mid f\rangle}{2}\right)^2} + \left(\frac{\langle f \mid g\rangle - \langle g \mid f\rangle}{2i}\right)^2$$

$$(\Delta A)^2(\Delta B^2) \geq \left(\left\langle\frac{[\hat{A},\hat{B}]}{2i}\right\rangle\right)^2$$

- For minimum uncertainty, we need
  - $|g\rangle = \alpha|f\rangle$ (with complex α). This gives us equality in the Schwarz Inequality.
  - $\langle f \mid g\rangle + \langle g \mid f\rangle = 0 \Rightarrow \alpha = i\lambda$

  And so $\left(\hat{B} - \langle\hat{B}\rangle\right)|\psi\rangle = i\lambda\left(\hat{A} - \langle\hat{A}\rangle\right)|\psi\rangle$
- For an eigenstate, the uncertainty of the operator is 0. To see why, consider $\langle\beta|\beta\rangle$ where $|\beta\rangle = (\hat{A} - a)|\alpha\rangle$. If $\langle\hat{A}^2\rangle = \langle\hat{A}\rangle^2$.

Quantum Dynamics

- **Time evolution** operator defined such that $|\psi, t\rangle = \hat{U}(t, t_0)|\psi, t_0\rangle$
  - **If $\hat{H}$ is independent of time**

    $$\boxed{\hat{U}(t, t_0) = e^{-i(t-t_0)\hat{H}/\hbar}}$$
  - **If $\hat{H}$ dependent on time but** $\left[\hat{H}(t), \hat{H}(t')\right] = 0$

    $$\boxed{\hat{U}(t, t_0) = e^{-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(\tilde{t})\mathrm{d}\tilde{t}}}$$

    [Eg: field with constant position and varying direction].
  - **If $\hat{H}$ dependent of time and** $\left[\hat{H}(t), \hat{H}(t')\right] \neq 0$, not 8.05
- For the first case above, insert the identity before and after $\hat{U}$ to find that

$$|\psi, 0\rangle = \sum_n C_n|n\rangle \Rightarrow |\psi, t\rangle = \sum_n C_n e^{-\frac{i}{\hbar}(t-t_0)E_n}|n\rangle$$

- We can view expectation values in two different ways

$$\overbrace{\left\langle \psi,0 \right| U^\dagger(t,0)}^{\boxed{\text{Schrodinger}} \atop {}_S\langle\psi,t|}\ \overbrace{\hat{A}}^{\hat{A}_S}\ \overbrace{U(t,0)\left|\psi,0\right\rangle}^{|\psi,t\rangle_S} = \underbrace{\left\langle \psi,0 \right|}_{{}_H\langle\psi|}\ \underbrace{U^\dagger(t,0)\hat{A}\,U(t,0)}_{\hat{A}_H}\ \underbrace{\left|\psi,0\right\rangle}_{|\psi\rangle_H \atop \boxed{\text{Heisenberg}}}$$

In the **Schrodinger Picture**, the **wavefunctions evolve with time** and **operators stay constant**. In the **Heisenberg picture**, the opposite is true.

- Schrodinger and Heisenberg operators are related by the **Heisenberg Equation of Motion**

$$\boxed{i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\hat{A}_H(t) = \left[\hat{A}_H(t), \hat{H}_H(t)\right] + i\hbar \underbrace{\dot{\hat{A}}_H(t)}_{=\hat{U}^\dagger \dot{\hat{A}}_S \hat{U}}}$$

This last term disappears if the Schrodinger operator does not vary with time, which is true most of the time. To **solve** this equation, find the **right-hand-side** and **integrate**. Taking **expectation values** of each side gives the **Ehrenfest Theorem**.

- A few notes
  - Changing picture does not change the form of commutators
  $$\left[\hat{A}_S, \hat{B}_S\right] = \hat{C}_S \Leftrightarrow \left[\hat{A}_H, \hat{B}_H\right] = \hat{C}_H$$
  - If $\left[\hat{H}(t), \hat{H}(t')\right] = 0$, then $\left[\hat{U}, \hat{H}\right] = 0$ and $\hat{H}_H = U^\dagger \hat{H}_S \hat{U} = \hat{H}_S$.
  - If $\left[\hat{H}, \hat{A}\right] = 0$, then A is a **conserved quantity**.

Two-State Systems

- The matrix element $\left\langle 2 \right| \hat{H} \left| 1 \right\rangle$ is a measure of the tunnelling probability 1 $\to$ 2.
- The **Hamiltonian** for a spin in a field $\boldsymbol{B}$ is
$$\hat{H} = -\gamma \boldsymbol{B} \cdot \hat{\boldsymbol{S}}$$
Where $\boldsymbol{S}$ is a vector containing the spin operators as its components. By using the **Heisenberg Equation of Motion** for S_x, S_y and S_z and integrating, we can show that any spin **precesses** about $\boldsymbol{B}$ with angular velocity $\omega = \gamma\left|B\right|$.

- Any **general two-state Hamiltonian** can be written as a sum of the **identity matrix** and the **Pauli matrices**, and so can be thought of as a **precessing spin**.
- Now, consider instead a system with a **constant field** in the z-direction, and a rotating field in the x–y plane ($\omega_0 = \gamma B_0$ and $\omega_1 = \gamma B_1$)
$$\hat{H}(t) = -\tfrac{\hbar}{2}\omega_0\hat{\sigma}_3 - \tfrac{\hbar}{2}\omega_1\left[\cos(\omega t)\hat{\sigma}_1 - \sin(\omega t)\hat{\sigma}_2\right]$$
Using the properties of the Pauli Matrices
$$\begin{aligned}\hat{H}(t) &= -\tfrac{\hbar}{2}\omega_0\hat{\sigma}_3 - \tfrac{\hbar}{2}\omega_1\exp\left(\tfrac{1}{2}i\omega t\sigma_3\right)\hat{\sigma}_1\exp\left(-\tfrac{1}{2}i\omega t\sigma_3\right) \\ &= -\exp\left(\tfrac{1}{2}i\omega t\sigma_3\right)\left[\tfrac{\hbar}{2}\omega_0\hat{\sigma}_3 + \tfrac{\hbar}{2}\omega_1\hat{\sigma}_1\right]\exp\left(-\tfrac{1}{2}i\omega t\sigma_3\right) \\ &= -\exp\left(\tfrac{1}{2}i\omega t\sigma_3\right)\left[\omega_0\hat{S}_z + \omega_1\hat{S}_x\right]\exp\left(-\tfrac{1}{2}i\omega t\sigma_3\right)\end{aligned}$$
So in other words, our Hamiltonian is **constant** in a **rotating frame**. So if the state is $\left|\psi_R(t)\right\rangle$ in the rotating frame, then in the lab frame, it is $\left|\psi(t)\right\rangle = \exp\left(\tfrac{1}{2}i\omega t\sigma_3\right)\left|\psi_R(t)\right\rangle$.
- Substitute $\left|\psi(t)\right\rangle = \exp\left(\tfrac{1}{2}i\omega t\sigma_3\right)\left|\psi_R(t)\right\rangle$ into the **LHS** of the SE to get
$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi_R(t)\right\rangle = \left[(\omega-\omega_0)\hat{S}_z - \omega_1\hat{S}_x\right]\left|\psi_R(t)\right\rangle$$
So in the rotating frame, there is **precession** about a **new field $\boldsymbol{B}_{\text{eff}}$**. Since, typically, $\omega_1 \ll \omega_2$, the possible options are as follows
  - ω **very different from** ω_0 – the field basically precesses around the z axis (ie: nearly not at all for a spin up).
  - $\omega \approx \omega_0$ – the field precesses around the x-axis at a frequency ω_1. Since our rotating frame is also moving around the z-axis, the spins spirals all the way down.
- **NMR** works as follows
  - We turn on a radio pulse with $\omega = \gamma_{\text{proton}}B_0$, for strength B_1 for a time $\Delta t = \pi / 2\gamma_{\text{proton}}B_1$. This brings the spin "down" and makes it maximally perpendicular to the z-axis.
  - We then switch the field off and look for radio emission of precessing spins at a frequency ω_0 resulting from such spins.

QM in Three-Dimensions

- Using all kinds of horribly complicated maths, we derive

$$\boldsymbol{L}^2 + \left(\boldsymbol{r}\cdot\boldsymbol{p}\right)^2 = \boldsymbol{r}^2\boldsymbol{p}^2 + i\hbar\boldsymbol{r}\cdot\boldsymbol{p} \Rightarrow \boldsymbol{p}^2 = \frac{(\boldsymbol{r}\cdot\boldsymbol{p})^2 - i\hbar\boldsymbol{r}\cdot\boldsymbol{p} + \boldsymbol{L}^2}{\boldsymbol{r}^2}$$

  Using this relation and even more complicated maths, we get

$$\boxed{\boldsymbol{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{\boldsymbol{L}^2}{2mr^2} + V(r)}$$

  And finally, using a last dose of complicated maths, we find that a **maximal set of commuting operators** for this system is

$$\boxed{\left\{H, \boldsymbol{L}^2, L_z\right\}}$$

- Start with eigenfunctions $\left|\ell, m\right\rangle$, and **assume** that

$$\boldsymbol{L}^2\left|\ell,m\right\rangle = \hbar^2\ell(\ell+1)\left|\ell,m\right\rangle$$
$$L_z\left|\ell,m\right\rangle = \hbar m\left|\ell,m\right\rangle$$

  Use horrible maths once again to get

$$\boxed{\boldsymbol{L}^2 = L_+L_- + L_z^2 - \hbar L_z}$$

  Then, derive facts about these as follows:

  - **Constraint on** ℓ

$$\left\langle \ell,m \mid \boldsymbol{L}^2 \mid \ell,m\right\rangle = \left\|L_i\left|\ell,m\right\rangle\right\| \geq 0$$
$$\left\langle \ell,m \mid \boldsymbol{L}^2 \mid \ell,m\right\rangle = \ell(\ell+1)\left\langle \ell,m \middle| \ell,m\right\rangle$$

    And so

$$\ell(\ell+1) \geq 0 \Rightarrow \boxed{\ell \geq 0}$$

  - **Constraint on** m

$$\left\langle \ell,m \mid L_-L_+ \mid \ell,m\right\rangle = \left\|L_+\left|\ell,m\right\rangle\right\| \geq 0$$
$$\left\langle \ell,m \mid L_+L_- \mid \ell,m\right\rangle = \left\langle \ell,m \mid \boldsymbol{L}^2 - L_z^2 - \hbar L_z \mid \ell,m\right\rangle$$

    And so

$$\ell(\ell+1) \geq m(m+1)$$

    When we have $\left|\ell, m_{\max}\right\rangle$, we have equality because $L_+\left|\ell,m_{\max}\right\rangle = 0$, and so $m_{\max} = \ell$. Doing the same with L_+L_-, we find

$$\boxed{-\ell \leq m \leq \ell}$$

  - **Action of ladder operators** – consider $\hat{\boldsymbol{L}}^2$ and $\hat{L}_z$ acting on $L_\pm\left|\ell,m\right\rangle$ to prove the lowering action. Write
$$L_\pm\left|\ell,m\right\rangle = C_\pm\left|\ell,m\pm 1\right\rangle$$
$$\left|C_\pm(\ell,m)\right|^2 = \left\langle \ell,m \mid L_\mp L_\pm \mid \ell,m\right\rangle$$
And then find $C_\pm$ by writing the product of ladder operators as above.

- To find $Y_{\ell m} = \left\langle \theta,\phi \middle| \ell,m\right\rangle$
  - Apply $\left\langle \theta,\phi\right|$ to the left of both sides of $L_z\left|\ell,m\right\rangle = \hbar m\left|\ell,m\right\rangle$
  - Separate variables, and get ϕ dependence directly.
  - Apply $\left\langle \theta,\phi\right|$ to the left of both sides of $L_+\left|\ell,\ell\right\rangle = 0$ to find $Y_{\ell\ell}$
  - Lower to find others.

  Note that $Y_{1,0}$ is a dumbbell, but $Y_{1,\pm 1}$ are doughnuts.
- Half–integer ℓ is impossible for spatial wavefunctions, because they can otherwise be lowered forever.
- The **parity operator** Π is defined by $\Pi\left|\boldsymbol{r}\right\rangle = \left|-\boldsymbol{r}\right\rangle$, and it is **hermitian** and **unitary**. It can be shown that
$$\Pi\left|\ell,m\right\rangle = \left(-1\right)^\ell\left|\ell,m\right\rangle$$
- Separating variables on the Schrodinger Equation gives
$$\boxed{\left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\hbar^2\ell(\ell+1)}{2mr^2} + V(r)\right)U(r) = EU(r)}$$
  Where
$$\boxed{\psi(\boldsymbol{r}) = \frac{1}{r}U(r)Y_{\ell m}(\theta,\phi)}$$
  And normalisation implies that
$$\int_0^\infty \left|U(r)\right|^2\,\mathrm{d}r = 1$$
- We can derive some asymptotic conditions on U:
  - If $U(r)\underset{r\to 0}{\longrightarrow} r^S$, then $S = \ell+1$, assuming that the potential is no more singular than $1/r$. Can be shown by solving the above and normalising, or by requiring H to be Hermitian.

- Assuming that V vanishes at infinity, then $U(r) \sim e^{\pm r\sqrt{-2mE/\hbar^2}}$, and depending on whether $E > 0$ or $E < 0$, we get planes waves or decaying exponentials.

- When "sketching" states:
  - The states starts off as $U(r) \sim r^{\ell+1}$
  - At ∞, we either have a sinusoidal function or a decaying exponential.
  - In between, we have **oscillatory behaviour**, where a **higher potential** (less energy) means a **higher amplitude** and a **longer wavelength**.
- The **super symmetric method**
  - We define pairs of related Hamiltonians, $H^{(1)} = \mathcal{A}^\dagger \mathcal{A}$, $H^{(2)} = \mathcal{A}\mathcal{A}^\dagger$. [Note that in each of the Hamiltonians, only the sign of the $\mathcal{W}'$ changes].
  - Important facts are that:
    1. $H^{(1)}$ and $H^{(2)}$ have the **same energy spectrum**, and if ϕ_n is an eigenstate of $H^{(1)}$, then $\mathcal{A}\phi_n$ is an eigenstate of $H^{(2)}$ with the **same eigenvalue**.
    2. There is usually some sort of relationship between $H^{(1)}$ and $H^{(2)}$.
    3. Only one of $H^{(1)}$ and $H^{(2)}$ can have a normalisable state with 0 energy.
  - So the tactic for these problems is
    - Use **3** to get a state of $H^{(1)}$, say.
    - Use **2** to get the next level state, but for $H^{(2)}$.
    - Use **1** to make that into a state of $H^{(1)}$
    - Rinse, lather, repeat…

Spin

- Eigenvalues of the Pauli matrices are
  - $\frac{\sqrt{2}}{2}(1,-1)$ and $\frac{\sqrt{2}}{2}(1,1)$ for σ_x.
  - $\frac{\sqrt{2}}{2}(-i,1)$ and $\frac{\sqrt{2}}{2}(1,-i)$ for σ_y.

- We can decompose any 2 by 2 matrix into

$$\boxed{\boldsymbol{M} = a_0 \boldsymbol{I} + \boldsymbol{a}\cdot\boldsymbol{\sigma}}$$

$$\left[a_0 = \tfrac{1}{2}\,\mathrm{tr}\left(\boldsymbol{M}\right) \qquad \boldsymbol{a} = \tfrac{1}{2}\,\mathrm{tr}\left(\boldsymbol{M}\boldsymbol{\sigma}\right)\right]$$

Addition of Angular Momenta

- Two angular momenta $\boldsymbol{L}$ and $\boldsymbol{L}'$, with $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{L}'$ can be described in two difference bases
  - $\hat{\boldsymbol{L}}^2, \hat{\boldsymbol{L}}'^2, \hat{L}_z, \hat{L}'_z$, and states are $\left|\ell, \ell', m_\ell, m'_\ell\right\rangle$
  - $\hat{\boldsymbol{L}}^2, \hat{\boldsymbol{L}}'^2, \hat{\boldsymbol{J}}^2, \hat{J}_z$, and states are $\left|\ell, \ell', J, M_j\right\rangle$
- It is useful to have the $\boldsymbol{L}\cdot\boldsymbol{L}'$ operator in the two bases

$$\boxed{\begin{gathered}\hat{\boldsymbol{L}}\cdot\hat{\boldsymbol{L}}' = \tfrac{1}{2}\left(\hat{\boldsymbol{J}}^2 - \hat{\boldsymbol{L}}^2 - \hat{\boldsymbol{L}}'^2\right) \\ \hat{\boldsymbol{L}}\cdot\hat{\boldsymbol{L}}' = \hat{L}_z\hat{L}'_z + \hat{L}_x\hat{L}'_x + \hat{L}_y\hat{L}'_y = \hat{L}_z\hat{L}'_z + \tfrac{1}{2}\left(\hat{L}_+\hat{L}'_- + \hat{L}_-\hat{L}'_+\right)\end{gathered}}$$

  In systems in which both bases are referred to in the Hamiltonian, it pays to stay in the $\hat{\boldsymbol{L}}^2, \hat{\boldsymbol{L}}'^2, \hat{L}_z, \hat{L}'_z$ basis and diagonalise the Hamiltonian.
- The transformations between the two bases are listed in tables, and are obtained roughly as follows:
  - **Start from the maximum J and M_j value** (trivially obtained by adding all the maximum states). Lower using the ladder operators.
  - **Find the state with $J - 1$ and $M_j = J - 1$** by using the fact it will be orthogonal to the **J, $M_j = J - 1$ state.** Lower.
  - **Rinse, lather, repeat…**

Identical Particles

- $\hat{P}_{ij}$ is the **exchange operator** – it exchanges *all* the labels i and j in a state. It is **Hermitian** and **unitary**, and can have **eigenvalues** + 1 (*bosons with integer spin*) or –1 (*fermions with half-integer spin*).

- *Constructing symmetric and antisymmetric wavefunctions*
  - Consider ***N*** **Fermions** (1, 2, …, N) which could be in any state $\alpha, \beta, \cdots, \nu$. The most general **antisymmetric linear combination** of these states is given by the **Slater Determinant**

$$\Psi(1,2,\cdots,N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} u_\alpha(1) & u_\alpha(2) & \cdots & u_\alpha(N) \\ u_\beta(1) & u_\beta(2) & \cdots & u_\beta(N) \\ \vdots & \vdots & & \vdots \\ u_\nu(1) & u_\nu(2) & \cdots & u_\nu(N) \end{vmatrix}$$

    Thus, if we're looking for a state in which we ***know*** one particle is in α, one in β and one in γ, we simply calculate the above. Notes:
    - Swapping one particle does exactly what is expected.
    - If *any* states are identical, the determinant goes to 0 and the wavefunction **cannot** be anti-symmetrised.
  - For ***N*** **bosons**, similar considerations apply, but with all the **alternating signs** in the **Slater determinant** changed to "positives".
- *General results* – if we have N particles and j possible states, then the number of different three particle states possible is
  - j^N if the particles are **distinguishable**.
  - ${}^{N+j-1}C_{j-1}$ if the particles are **indistinguishable bosons**. [Placing $j-1$ barriers between $N+j-1$ states].
  - ${}^{j}C_{N}$ if the particles are **indistinguishable fermions**.
- *Spatial and spin parts*
  - It is also possible to factor a wavefunction into **spin** and **spatial** wavefunctions. It is the **product of both** that has to satisfy appropriate symmetry.
  - We can **individually symmetrise/antisymmetrise** each part using the tactics above.
  - In general, for the $\ell \otimes \ell'$ spin case, states with resulting **even** J will be even, and states with resulting **odd** J will be odd.
- *Correlation and exchange forces*

- **Symmetric** wavefunctions result in particles appearing to "**attract**" each other, and vice-versa.
- Thus, energy levels in the Helium molecule are *lower* when spins are *aligned* (symmetric) because the spatial part then has to be *antisymmetric*, which results in less repulsion.
- Similarly, bonds are caused by *antisymmetric* spins, which then cause the electrons to attract.